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Symmetries and their Lie algebra properties for the higher-order Burgers equations

Tao Sun† and Wei Wang

Physics Department, Liaoning Normal University, Dalian, People's Republic of China

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Abstract. A new strong symmetry, two groups of symmetries and their Lie algebra properties for the higher-order Burgers equations are presented.

1. Introduction

In the preceding paper (Tao Sun 1989) we discussed the symmetries and Lie algebra properties of the Burgers equation. A new class of symmetries has been found. It is known that the Burgers equation has two strong symmetries Φ and Ψ (Tian Chou 1987)

$$\Phi = \mathrm{D} + u + u_x \mathrm{D}^{-1} \tag{1.1}$$

$$\Psi = 2t\Phi + x + \mathrm{D}^{-1} \tag{1.2}$$

and three groups of symmetries

$$K_n = \Phi^n K_0 \qquad K_0 = u_x \tag{1.3}$$

$$\tau_n = \Phi^n \tau_0 \qquad \tau_0 = 2tu_x + 1 \tag{1.4}$$

$$u_n(\varepsilon) = \Psi^n \mu_0(\varepsilon) \qquad \mu_0(\varepsilon) = (u - \varepsilon)\exp(-\mathrm{D}^{-1}u + \varepsilon x + \varepsilon^2 t) \tag{1.5}$$

where $n = 0, 1, 2, \ldots$.

In this work we discuss the symmetries of the higher-order Burgers equation

$$u_t = K_l = \Phi^l u_x \qquad l = 1, 2, \ldots \tag{1.6}$$

where Φ is given in (1.1). To our knowledge, very little is known about the symmetries of this set of equations, except that Φ has been proved to be a strong symmetry of them (Tian Chou 1987). In the following we show that there exists a strong symmetry Ψ_l and two groups of symmetries $\kappa_n^{(l)}(\varepsilon)$ and $\mu_n^{(l)}(\varepsilon)$. Their Lie algebra properties have also been identified.

2. Strong symmetry

Theorem 2.1. The operator

$$\Psi_l = (l+1)t\Phi^l + x + \mathrm{D}^{-1} \qquad l = 1, 2, \ldots \tag{2.1}$$

is a strong symmetry for the l-order Burgers equation (1.6).

† Present address: Department of Physics, McGill University, 3600 University Street, Montreal, Quebec, Canada H3A 2T8.

Proof. From (2.1) we have

$$\mathrm{d}\Psi_l/\mathrm{d}t = (l+1)\Phi^l + (l+1)t\,\mathrm{d}\Phi^l/\mathrm{d}t$$

$$[K_l', \Psi_l] = (l+1)t[K_l', \Phi^l] + [K_l', x + \mathrm{D}^{-1}].$$

Since Φ is a strong symmetry, Φ^l is also a strong symmetry (Tian Chou 1987), i.e.

$$\mathrm{d}\Phi^l/\mathrm{d}t = [K_l', \Phi^l].$$

Next, notice that (Tian Chou 1987):

$$[K_l', x + \mathrm{D}^{-1}] = (l+1)\Phi^l \qquad l = 0, 1, 2, \ldots.$$

Therefore

$$\mathrm{d}\Psi_l/\mathrm{d}t = [K_l', \Psi_l]$$

which means that Ψ_l is a strong symmetry of the l-order Burgers equation (1.6).

Theorem 2.2.

$$[\Phi^n, \Psi_l] = n\Phi^{n-1} \qquad n = 1, 2, \ldots; \quad l = 1, 2, \ldots. \tag{2.2}$$

The proof is by induction of n. For $n = 1$, we have

$$[\Phi, \Psi] = \mathrm{I}. \tag{2.2'}$$

Lemma 2.1.

$$\Phi'[(x + \mathrm{D}^{-1})a] = (x + \mathrm{D}^{-1})\Phi'[a] + \mathrm{D}^{-1}a + a\,\mathrm{D}^{-1} \tag{2.3}$$

is valid for any function a.

Proof.

$$\Phi'[(x + \mathrm{D}^{-1})a] = xa + (\mathrm{D}^{-1}a) + 2a\,\mathrm{D}^{-1} + xa_x\,\mathrm{D}^{-1}$$
$$= (x + \mathrm{D}^{-1})\Phi'[a] + \mathrm{D}^{-1}a + a\mathrm{D}^{-1}.$$

Lemma 2.2.

$$\{[(\mathrm{D}^{-1}a) + a\mathrm{D}^{-1}]\Phi^n + (\Phi^n)'[a]\}b - \{[(\mathrm{D}^{-1}b) + b\,\mathrm{D}^{-1}]\Phi^n + (\Phi^n)'[b]\}a = 0 \qquad n = 0, 1, 2, \ldots \tag{2.4}$$

is valid for any functions a and b.

Lemma 2.3.

$$(X + \mathrm{D}^{-1})\{(\Phi^n)'[a]b - (\Phi^n)'[b]a\} = (\Phi^n)'[(x + \mathrm{D}^{-1})a]b - (\Phi^n)'[(x + \mathrm{D}^{-1})b]a \qquad n = 1, 2, \ldots \tag{2.5}$$

is valid for any functions a and b.

The proofs of lemmas (2.2) and (2.3) are by induction on n.

Theorem 2.3. Ψ_l is a hereditary symmetry, i.e.

$$\Psi_l'[\Psi_l a]b - \Psi_l'[\Psi_l b]a = \Psi_l\{\Psi_l'[a]b - \Psi_l'[b]a\} \qquad l = 1, 2, \ldots \tag{2.6}$$

is valid for any functions a and b.

Proof. Given (2.1), what we need to do is just prove

$$(\Phi^l)'[\Psi_l a]b - (\Phi^l)'[\Psi_l b]a = \Psi_l\{(\Phi^l)'[a]b - (\Phi^l)'[b]a\}. \tag{2.6a}$$

Since Φ is a hereditary symmetry, according to Tao Sun (1989, lemma 2.4) and (2.5), we obtain

$$\begin{aligned}
&\Psi_l\{(\Phi^l)'[a]b - (\Phi^l)'[b]a\} \\
&\quad = (l+1)t\Phi^l\{(\Phi^l)'[a]b - (\Phi^l)'[b]a\} + (x + \mathrm{D}^{-1})\{(\Phi^l)'[a]b - (\Phi^l)'[b]a\} \\
&\quad = (\Phi^l)'[(l+1)t\Phi^l a]b - (\Phi^l)'[(l+1)t\Phi^l b]a \\
&\qquad + (\Phi^l)'[(x + \mathrm{D}^{-1})a]b - (\Phi^l)'[(x + \mathrm{D}^{-1})b]a \\
&\quad = (\Phi^l)'[\Psi_l a]b - (\Phi^l)'[\Psi_l b]a.
\end{aligned}$$

3. Symmetries

Lemma 3.1.

$$K_n'[1] = (n+1)K_{n-1} \qquad n = 1, 2, \ldots. \tag{3.1}$$

The proof is by induction on n.

Lemma 3.2.

$$\mathrm{d}K_n/\mathrm{d}t = K_m'[K_n] \qquad m = 1, 2, \ldots; \quad n = 0, 1, 2, \ldots. \tag{3.2}$$

Proof. Equation (3.2) implies that K is a symmetry of the equation $u_t = K_m$. Since the strong symmetry Φ maps a symmetry to a symmetry, it is sufficient to prove (3.2) for $n = 0$, i.e.

$$\mathrm{d}K_0/\mathrm{d}t = K_m'[K_0] \qquad m = 1, 2, \ldots. \tag{3.2a}$$

Equation (3.2a) is obviously true for $m = 1$. Assume that it is established when $m = k - 1$ and let us prove it for $m = k$. In fact

$$\begin{aligned}
K_k'[K_0] &= \Phi'[K_0]K_{k-1} + \Phi K_{k-1}'[K_0] \\
&= K_0 K_{k-1} + (\mathrm{D}K_0)(\mathrm{D}^{-1} K_{k-1}) + \Phi\, \mathrm{D}K_{k-1} \\
&= (\mathrm{D}^2 + 2u_x + u\mathrm{D} + u_{xx}\, \mathrm{D}^{-1})K_{k-1} = \mathrm{D}\, K_k = \mathrm{d}K_0/\mathrm{d}t.
\end{aligned}$$

Let

$$\begin{aligned}
\kappa_0^{(l)}(\alpha, \beta) &= [\alpha(l+1)t + \beta]\Phi^{l-1}u_x + \alpha \\
&= [\alpha(l+1)t + \beta]K_{l-1} + \alpha
\end{aligned} \tag{3.3}$$

$$u_0^{(l)}(\varepsilon) = (u - \varepsilon)\exp(-D^{-1}u + \varepsilon x + \varepsilon^{l+1}t) \tag{3.4}$$

$$\begin{aligned}
\kappa_n^{(l)}(\alpha, \beta) &= \Phi^n \kappa_0^{(l)}(\alpha, \beta) \\
&= [\alpha(l+1)t + \beta]K_{l+n-1} + \alpha\Phi^n 1
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&u_n^{(l)}(\varepsilon) = \Psi_l^n \mu_0^{(l)}(\varepsilon) \\
&l = 1, 2, \ldots \qquad n = 0, 1, 2, \ldots
\end{aligned} \tag{3.6}$$

where α, β and ε are arbitrary constants independent of time and space.

Theorem 3.1. $\kappa_n^{(l)}(\alpha,\beta)$ and $\mu_n^{(l)}(\varepsilon)$ are symmetries for the l-order Burgers equation (1.6), i.e.

$$\mathrm{d}\kappa_n^{(l)}(\alpha,\beta)/\mathrm{d}t = K_l'[\kappa_n^{(l)}(\alpha,\beta)] \tag{3.7}$$

$$\mathrm{d}\mu_n^{(l)}(\varepsilon)/\mathrm{d}t = K_l'[\mu_n^{(l)}(\varepsilon)]. \tag{3.8}$$

Proof. In a similar way to the proof of lemma 3.2, what we need to do is just prove

$$\mathrm{d}\kappa_0^{(l)}(\alpha,\beta)/\mathrm{d}t = K_l'[\kappa_0^{(l)}(\alpha,\beta)] \tag{3.7$'$}$$

$$\mathrm{d}\mu_0^{(l)}(\varepsilon)/\mathrm{d}t = K_l'[\mu_0^{(l)}(\varepsilon)]. \tag{3.8$'$}$$

From (3.3) we have

$$\mathrm{d}\kappa_0^{(l)}(\alpha,\beta)/\mathrm{d}t = \alpha(l+1)K_{l-1} + [\alpha(l+1)t+\beta]\,\mathrm{d}K_{l-1}/\mathrm{d}t$$

$$K_l'[\kappa_0^{(l)}(\alpha,\beta)] = [\alpha(l+1)t+\beta]K_l'[K_{l-1}] + \alpha K_l'[1].$$

According to (3.1) and (3.2), (3.7′) is established.

For the case of $\mu_0^{(l)}(\varepsilon)$, (3.8′) is obviously valid when $l=1$ (Tao Sun 1989). Assume it is valid for $l=k-1$, we prove that it is established for $l=k$. In fact

$$\begin{aligned}
K_k'[\mu_0^{(k)}(\varepsilon)] &= (\Phi K_{k-1})'[\mu_0^{(k)}(\varepsilon)]\\
&= \Phi'[\mu_0^{(k)}(\varepsilon)]K_{k-1} + \Phi K_{k-1}'[\mu_0^{(k)}(\varepsilon)]\\
&= \mu_0^{(k)}(\varepsilon)K_{k-1} + [\mathrm{D}\mu_0^{(k)}(\varepsilon)](\mathrm{D}^{-1}K_{k-1}) + \exp[\varepsilon^k(\varepsilon-1)t]\Phi K_{k-1}'[\mu_0^{(k-1)}(\varepsilon)]\\
&= \{K_k + (u-\varepsilon)[-(\mathrm{D}^{-1}K_k)+\varepsilon^{k+1}]\}\exp(-\mathrm{D}^{-1}u+\varepsilon x+\varepsilon^{k+1}t)\\
&= \mathrm{d}\,\mu_0^{(k)}(\varepsilon)/\mathrm{d}t
\end{aligned}$$

which implies (3.8).

4. Preliminary theorems

Lemma 4.1.1.

$$\Phi\mu_n^{(l)}(\varepsilon) = \varepsilon\mu_n^{(l)}(\varepsilon) + n\mu_{n-1}^{(l)}(\varepsilon) \qquad l=1,2,\ldots;\quad n=0,1,2,\ldots. \tag{4.1}$$

The proof is by induction on n.

Theorem 4.1.

$$\Phi^m\mu_n^{(l)}(\varepsilon) = \sum_{r=0}^{m} C_m^r[n!/(n-r)!]\varepsilon^{m-r}\mu_{n-r}^{(l)}(\varepsilon) \tag{4.2}$$

$$l=1,2,\ldots \qquad m=1,2,\ldots \qquad n=0,1,2,\ldots.$$

Proof. From (4.1), (4.2) is valid for $m=1$. Assume that it is true when $m=k-1$, we prove it for $m=k$. In fact

$$\begin{aligned}
\Phi^k\mu_n^{(l)}(\varepsilon) &= \Phi\Phi^{k-1}\mu_n^{(l)}(\varepsilon)\\
&= \sum_{r=0}^{k-1} C_{k-1}^r[n!/(n-r)!]\varepsilon^{k-r-1}[\varepsilon\mu_n^{(l)}(\varepsilon)+(n-r)\mu_{n-r-1}^{(l)}(\varepsilon)]\\
&= \sum_{r=0}^{k} C_k^r[n!/(n-r)!]\varepsilon^{k-r}\mu_{n-r}^{(l)}(\varepsilon).
\end{aligned}$$

Lemma 4.2.1.

$$\mu_n^{(l)}(\varepsilon)' + D^{-1}\,\mu_n^{(l)}(\varepsilon) + \mu_n^{(l)}(\varepsilon)\,D^{-1} = 0$$
$$l = 1, 2, \ldots \qquad n = 0, 1, 2, \ldots . \tag{4.3}$$

The proof is by induction on n.

Theorem 4.2.

$$\Phi'[\mu_n^{(l)}(\varepsilon)] = [\mu_n^{(l)}(\varepsilon)', \Phi] \qquad l = 1, 2, \ldots; \quad n = 0, 1, 2, \ldots . \tag{4.4}$$

Proof. For $n = 0$, $\mu_0^{(l)}(\varepsilon) = \exp[(\varepsilon^{l+1} - \varepsilon^2)t]\mu_0(\varepsilon)$, so (4.4) is true by lemma 4.3 of Tao Sun (1989). Assuming it is established for $n = k-1$, we prove it for $n = k$. In fact, by (2.3), (4.2)and (4.3)

$$\begin{aligned}
[\mu_k^{(l)}(\varepsilon)', \Phi] &= [(\Psi_l \mu_{k-1}^{(l)}(\varepsilon))', \Phi] \\
&= (l+1)t \sum_{r=0}^{l} C_l^r[(k-1)!/(k-r-1)!]\varepsilon^{l-r}[\mu_{k-r-1}^{(l)}(\varepsilon)', \Phi] \\
&\quad + [(x + D^{-1})\mu_{k-1}^{(l)}(\varepsilon)', \Phi] \\
&= \Phi'[(l+1)t\Phi^l \mu_{k-1}^{(l)}(\varepsilon)] + (x + D^{-1})\Phi'[\mu_{k-1}^{(l)}(\varepsilon)] - \mu_{k-1}^{(l)}(\varepsilon)' \\
&= \Phi'[\mu_k^{(l)}(\varepsilon)] - [\mu_{k-1}^{(l)}(\varepsilon)' + D^{-1}\,\mu_{k-1}^{(l)}(\varepsilon) + \mu_{k-1}^{(l)}(\varepsilon)\,D^{-1}] \\
&= \Phi'[\mu_k^{(l)}(\varepsilon)].
\end{aligned}$$

Lemma 4.3.1.

$$(\Phi^n)'[\mu_0^{(l)}(\varepsilon)] = [\mu_0^{(l)}(\varepsilon), \Phi^n] \qquad l = 1, 2, \ldots; \quad n = 1, 2, \ldots . \tag{4.5}$$

The proof is by induction on n.

Lemma 4.3.2.

$$[\mu_0^{(l)}(\varepsilon)', x + D^{-1}] = 0 \qquad l = 1, 2, \ldots . \tag{4.6}$$

Proof. It is easy to check that (4.6) is established.

Lemma 4.3.3.

$$\Psi_l'[\mu_0^{(l)}(\varepsilon)] = [\mu_0^{(l)}(\varepsilon)', \Psi_l]. \tag{4.7}$$

Proof. Since

$$\Psi_l'[\mu_0^{(l)}(\varepsilon)] = (l+1)t(\Phi^l)'[\mu_0^{(l)}(\varepsilon)]$$
$$[\mu_0^{(l)}(\varepsilon)', \Psi_l] = (l+1)t[\mu_0^{(l)}(\varepsilon)', \Phi^l] + [\mu_0^{(l)}(\varepsilon)', x + D^{-1}]$$

according to (4.5) and (4.6), (4.7) is valid.

Theorem 4.3.

$$\Psi_l'[\mu_n^{(l)}(\varepsilon)] = [\mu_n^{(l)}(\varepsilon)', \Psi_l] \qquad n = 0, 1, 2, \ldots . \tag{4.8}$$

Proof. Since Ψ_l is a hereditary symmetry, from (3.6) and (4.7), (4.8) is established.

Theorem 4.4.

$$[K_m, \Phi^n 1] = (m+1)K_{m+n-1} \qquad m=1,2,\ldots \qquad n=0,1,2,\ldots. \tag{4.9}$$

Proof. From (3.1), (4.9) is valid for $n=0$. Assume it is valid for $n=k-1$; we prove it for $n=k$. In fact

$$\begin{aligned}[K_m, \Phi^k 1] &= K'_m[\Phi^k 1] - (\Phi^k 1)'[K_m] \\ &= K'_m[\Phi^k 1] - \Phi'[K_m]\Phi^{k-1}1 - \Phi(\Phi^{k-1}1)'[K_m].\end{aligned}$$

Since $\Phi'[K_m] = [K'_m, \Phi]$ (Tian Chou 1987), we obtain

$$[K_m, \Phi^k 1] = \Phi[K_m, \Phi^{k-1}1] = (m+1)K_{m+k-1}.$$

Lemma 4.5.1.

$$[\Phi^m 1, 1] = m\Phi^{m-1}1 \qquad m=1,2,\ldots. \tag{4.10}$$

The proof is by induction on m.

Theorem 4.5.

$$[\Phi^m 1, \Phi^n 1] = (m-n)\Phi^{m+n-1}1 \qquad m=1,2,\ldots, \qquad n=0,1,2,\ldots. \tag{4.11}$$

Proof. From (4.10), (4.11) is valid for $n=0$. Assume it is true for $n=k-1$; we prove it for $n=k$. In fact

$$\begin{aligned}[\Phi^m 1, \Phi^k 1] &= (\Phi^m 1)'[\Phi^k 1] - [\Phi^k 1]'[\Phi^m 1] \\ &= (\Phi^m)'[\Phi^k 1]1 - \Phi'[\Phi^m 1]\Phi^{k-1}1 - \Phi(\Phi^{k-1})'[\Phi^m 1]1.\end{aligned}$$

Since Φ is a hereditary symmetry, we have (Tao Sun 1989)

$$(\Phi^m)'[\Phi^k 1]1 - \Phi'[\Phi^m 1]\Phi^{k-1}1 = \Phi(\Phi^m)'[\Phi^{k-1}1]1 - \Phi^{m+k-1}1$$

then

$$\begin{aligned}[\Phi^m 1, \Phi^k 1] &= \Phi\{(\Phi^m)'[\Phi^{k-1}1]1 - (\Phi^{k-1})'[\Phi^m 1]1\} - \Phi^{m+k-1}1 \\ &= (m-k)\Phi^{m+k-1}1.\end{aligned}$$

which implies (4.11).

Lemma 4.6.1.

$$\Psi'_l[1] = l(l+1)t\Phi^{l-1} \qquad l=1,2,\ldots. \tag{4.12}$$

Proof. Since

$$\Psi_l'[1] = (l+1)t(\Phi^l)'[1]$$

what we need to do is just prove

$$(\Phi^l)'[1] = l\Phi^{l-1}. \tag{4.12$'$}$$

The proof of (4.12′) is by induction on l.

Lemma 4.6.2.

$$[1, \mu_n^{(l)}(\varepsilon)] = (x + D^{-1})\mu_n^{(l)}(\varepsilon) \qquad n = 0, 1, 2, \ldots. \tag{4.13}$$

The proof is by induction on n.

Theorem 4.6.

$$\begin{gathered}[\Phi^m 1, \mu_n^{(l)}(\varepsilon)] = \Phi^m(x + D^{-1})\mu_n^{(l)}(\varepsilon) \\ m = 0, 1, 2, \ldots \qquad n = 0, 1, 2, \ldots.\end{gathered} \tag{4.14}$$

Proof. By (4.13), (4.14) is valid for $m = 0$. Inducting on m, we have

$$\begin{aligned}[\Phi^k 1, \mu_n^{(l)}(\varepsilon)] &= (\Phi^k 1)'[\mu_n^{(l)}(\varepsilon)] - \mu_n^{(l)}(\varepsilon)'[\Phi^k 1] \\ &= \Phi'[\mu_n^{(l)}(\varepsilon)]\Phi^{k-1}1 + \Phi(\Phi^{k-1}1)'[\mu_n^{(l)}(\varepsilon)] - \mu_n^{(l)}(\varepsilon)'[\Phi^k 1] \\ &= \Phi[\Phi^{k-1}1, \mu_n^{(l)}(\varepsilon)] = \Phi^k(x + D^{-1})\mu_n^{(l)}(\varepsilon).\end{aligned}$$

Lemma 4.7.1.

$$\begin{gathered}[K_0, \mu_n^{(l)}(\varepsilon)] = \Phi\mu_n^{(l)}(\varepsilon) \\ l = 1, 2, \ldots \quad n = 0, 1, 2, \ldots.\end{gathered} \tag{4.15}$$

The proof is by induction on n.

Theorem 4.7.

$$[K_m, \mu_n^{(l)}(\varepsilon)] = \Phi^{m+1}\mu_n^{(l)}(\varepsilon) \tag{4.16}$$

$$l = 1, 2, \ldots \qquad m = 0, 1, 2, \ldots \qquad n = 0, 1, 2, \ldots.$$

Proof. From (4.15), (4.16) is valid for $m = 0$. Assume it is true for $m = k - 1$, we prove it for $m = k$. In fact, by (4.4),

$$\begin{aligned}[K_k, \mu_n^{(l)}(\varepsilon)] &= K_k'[\mu_n^{(l)}(\varepsilon)] - \mu_n^{(l)}(\varepsilon)'[K_k] \\ &= \Phi'[\mu_n^{(l)}(\varepsilon)]K_{k-1} + \Phi K_{k-1}'[\mu_n^{(l)}(\varepsilon)] - \mu_n^{(l)}(\varepsilon)'[K_k] \\ &= \Phi[K_{k-1}, \mu_n^{(l)}(\varepsilon)] = \Phi^{k+1}\mu_n^{(l)}(\varepsilon).\end{aligned}$$

5. Lie algebra

Theorem 5.1.

$$[\kappa_m^{(l)}(\alpha, \beta), \kappa_n^{(l)}(\alpha', \beta')] = \kappa_{m+n-1}^{(l)}(A, B)$$

where

$$A = \alpha\alpha'(m-n) \tag{5.1}$$
$$B = (\alpha'\beta - \alpha\beta')l + \alpha'\beta m - \alpha\beta' n$$
$$l = 1, 2, \ldots \qquad m = 0, 1, 2, \ldots \qquad n = 0, 1, 2, \ldots$$

and α, β, α' and β' are constants.

Proof. From (4.9), (4.11) and the relation $[K_m, K_n] = 0$ (Tian Chou 1987), we have

$$\begin{aligned}[\kappa_m^{(l)}(\alpha, \beta), \kappa_n^{(l)}(\alpha', \beta')]& \\ &= [\alpha(l+1)t+\beta][\alpha'(l+1)t+\beta'][K_{l+m-1}, K_{l+n-1}] \\ &\quad + \alpha'[\alpha(l+1)t+\beta][K_{l+m-1}, \Phi^n 1] \\ &\quad + \alpha[\alpha'(l+1)t+\beta'][\Phi^m l, K_{l+n-1}] + \alpha\alpha'[\Phi^m 1, \Phi^n 1] \\ &= [A(l+1)t+B]K_{l+m+n-2} + A\Phi^{m+n-1}1 = \kappa_{m+n-1}^{(l)}(A, B)\end{aligned}$$

where constants A and B are given in (5.1).

Theorem 5.2.

$$\begin{aligned}&[\mu_m^{(l)}(\varepsilon), \mu_n^{(l)}(\varepsilon')] = 0 \\ &l = 1, 2, \ldots \qquad m = 0, 1, 2, \ldots \qquad n = 0, 1, 2, \ldots .\end{aligned} \tag{5.2}$$

Proof. It is not difficult to check that

$$[\mu_0^{(l)}(\varepsilon), \mu_0^{(l)}(\varepsilon')] = 0 \qquad [\mu_0^{(l)}(\varepsilon), \mu_1^{(l)}(\varepsilon')] = 0.$$

Furthermore, (5.2) is evidently established because of (4.8).

Theorem 5.3.

$$\begin{aligned}[\kappa_m^{(l)}(\alpha, \beta), \mu_n^{(l)}(\varepsilon)]& \\ &= \alpha \sum_{r=0}^{m} C_m^r[(n+1)!/(n-r+1)!]\varepsilon^{m-r}\mu_{n-r+1}^{(l)}(\varepsilon) \\ &\quad + \beta \sum_{r=0}^{l+m} C_{l-m}^r[n!/(n-r)!]\varepsilon^{l+m-r}\mu_{n-r}^{(l)}(\varepsilon) \\ &l = 1, 2, \ldots \qquad m = 0, 1, 2, \ldots \qquad n = 0, 1, 2, \ldots .\end{aligned} \tag{5.3}$$

Proof. From (4.14) and (4.16), we can obtain

$$\begin{aligned}[\kappa_m^{(l)}(\alpha, \beta), \mu_n^{(l)}(\varepsilon)] &= [\alpha(l+1)t+\beta][K_{l+m-1}, \mu_n^{(l)}(\varepsilon)]+\alpha[\Phi^m 1, \mu_n^{(l)}(\varepsilon)] \\ &= \alpha\Phi^m\mu_{n+1}^{(l)}(\varepsilon)+\beta\Phi^{l+m}\mu_n^{(l)}(\varepsilon).\end{aligned}$$

According to (4.2), (5.3) is established.

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